Unbounded Anisotropy Formulation for the Elliptic Representation of the Boltzmann Equation

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The elliptic representation of the Boltzmann equation[1] has been shown to provide a useful approximation for the determination of the distribution function by making some heuristic assumptions about its angular dependence, and is valid under extremes of anisotropy. Subsequently, it has been shown[2] that an alternate expression of this approximation, explicitly in terms of anisitropy, can be derived.

For the sake of completeness, this anisotropic-based expression is, in the general case, shown here:

$$\begin{aligned}
\frac{\partial \vec{\mathbf{X}}}{\partial t} &+ \nabla \cdot \left(v G(X) \vec{\mathbf{X}} \vec{\mathbf{X}} \right) + v \vec{\mathbf{X}} \cdot \nabla \vec{\mathbf{X}} \\
&- \frac{\partial}{\partial u} \left(G(X) v \left(\vec{\mathbf{E}} \cdot \vec{\mathbf{X}} \right) \vec{\mathbf{X}} \right) - v \left(\vec{\mathbf{E}} \cdot \vec{\mathbf{X}} \right) \frac{\partial \vec{\mathbf{X}}}{\partial u} = \\
&- G(X) \frac{v}{n} \vec{\mathbf{X}} \vec{\mathbf{X}} \cdot \nabla n - \frac{v}{n} \nabla \left(n H(X) \right) \\
&+ G(X) \left(\vec{\mathbf{E}} \cdot \vec{\mathbf{X}} \right) \vec{\mathbf{X}} \frac{1}{n} \frac{\partial (vn)}{\partial u} + \frac{v}{n} \vec{\mathbf{E}} \frac{\partial (n H(X))}{\partial u} \\
&+ \frac{v}{2u} J(X) \left(\left(\vec{\mathbf{E}} \cdot \vec{\mathbf{X}} \right) \vec{\mathbf{X}} - X^2 \vec{\mathbf{E}} \right) + \left(\frac{\delta \vec{\mathbf{X}}}{\delta t} \right)_c
\end{aligned} \tag{1}$$

where:

$$G(X) = \left(\frac{1}{2X^2} \left(3\frac{X}{\gamma} - 1 \right) - 1 \right) \tag{2}$$

$$H(X) = \frac{1}{2} \left(1 - \frac{X}{\gamma} \right) \tag{3}$$

$$J(X) = \frac{1}{2X^2} \left(3\frac{X}{\gamma} - 1 \right) \tag{4}$$

and $\vec{\mathbf{X}} = \vec{\mathbf{\Gamma}}/n = \vec{\mathbf{f}}_1/(2f_0)$ is the quantitative anisotropy vector ($X = |\vec{\mathbf{X}}|$). This formulation will be referred to as the "X" formulation, and has advantages as a basis for discretization for the suppression of numerical artifacts. In particular, the generation of high-spatial-frequency oscillations in a low-dissipation time-dependent numerical scheme is largely mitigated by discretization according to the "X" formulation.

The main difficulty with the X formulation is that the range of $|\vec{\mathbf{X}}|$ is limited to $[0, 1]$. The inevitable truncation error of discretized numerical schemes will lead to violations of these limits. That is, although the continuum equations serve to limit the extent of $|\vec{\mathbf{X}}|$, computationally a limit is difficult to enforce. Certainly artificial limits can be placed, and are somewhat justified by knowledge of the properties of $\vec{\mathbf{X}}$, but these would present a deviation from the otherwise uniform treatment of the governing equation.

A more pleasing technique would be to seek yet another dependent variable with infinite range to represent the limited range of $|\vec{\mathbf{X}}|$. One such transformation is as follows. Define:

$$\vec{\mathbf{Y}} = \frac{\vec{\mathbf{X}}}{\sqrt{1 - X^2}} \tag{5}$$

so that:

$$\vec{\mathbf{X}} = \frac{\vec{\mathbf{Y}}}{\sqrt{1 + Y^2}} \tag{6}$$

Thus it is very easy to transform back and forth between $\vec{\mathbf{X}}$ and $\vec{\mathbf{Y}}$. The following identities are all easily derived:

$$\frac{Y}{X}\left(1 + \vec{\mathbf{Y}}\vec{\mathbf{Y}}\cdot\right)\frac{\partial \vec{\mathbf{X}}}{\partial s} = \frac{\partial \vec{\mathbf{Y}}}{\partial s} \tag{7}$$

where s is *any* independent variable.

$$\frac{Y}{X}\left(1 + \vec{\mathbf{Y}}\vec{\mathbf{Y}}\cdot\right)\vec{\mathbf{X}} = (1 + Y^2)\vec{\mathbf{Y}} \tag{8}$$

$$\frac{Y}{X}\left(1 + \vec{\mathbf{Y}}\vec{\mathbf{Y}}\cdot\right)\vec{\mathbf{E}} = \frac{Y}{X}\vec{\mathbf{E}} + \frac{Y}{X}(\vec{\mathbf{Y}} \cdot \vec{\mathbf{E}})\vec{\mathbf{Y}} \tag{9}$$

and these make it easy to transform Equation 1 into a transport equation for $\vec{\mathbf{Y}}$:

$$\begin{aligned}
\frac{\partial \vec{\mathbf{Y}}}{\partial t} &+ v\vec{\mathbf{X}} \cdot \nabla\vec{\mathbf{Y}} + \nabla \cdot \left(vG(X)\vec{\mathbf{X}}\vec{\mathbf{Y}}\right) \\
&- v\left(\vec{\mathbf{E}} \cdot \vec{\mathbf{X}}\right)\frac{\partial \vec{\mathbf{Y}}}{\partial u} - \frac{\partial}{\partial u}\left(vG(X)\left(\vec{\mathbf{E}} \cdot \vec{\mathbf{X}}\right)\vec{\mathbf{Y}}\right) = \\
&Y^2\vec{\mathbf{Y}}\frac{\partial}{\partial u}\left(vG(X)\left(\vec{\mathbf{E}} \cdot \vec{\mathbf{X}}\right)\right) \\
&- \frac{v}{n}G(X)(1 + Y^2)\vec{\mathbf{Y}}\vec{\mathbf{X}} \cdot \nabla n - \frac{v}{n}\frac{Y}{X}\left(1 + \vec{\mathbf{Y}}\vec{\mathbf{Y}}\cdot\right)\nabla(nH(X)) \\
&- vY^2\vec{\mathbf{Y}}\nabla \cdot \left(G(X)\vec{\mathbf{X}}\right) \\
&+ (1 + Y^2)G(X)\left(\vec{\mathbf{E}} \cdot \vec{\mathbf{X}}\right)\vec{\mathbf{Y}}\frac{1}{n}\frac{\partial(vn)}{\partial u}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{v}{n}\frac{\partial(nH(X))}{\partial u}\frac{Y}{X}\left(\vec{\mathbf{E}}+\vec{\mathbf{Y}}\left(\vec{\mathbf{Y}}\cdot\vec{\mathbf{E}}\right)\right) \\
&+ \quad \frac{v}{2u}J(X)\left(\left(\vec{\mathbf{E}}\cdot\vec{\mathbf{X}}\right)\vec{\mathbf{Y}} - XY\vec{\mathbf{E}}\right) + \left(\frac{\delta\vec{\mathbf{Y}}}{\delta t}\right)_c
\end{aligned} \tag{10}
$$

If the approximation $\left(\frac{\delta\vec{\mathbf{\Gamma}}}{\delta t}\right)_c = -\nu\vec{\mathbf{\Gamma}}$ is used, although it is not strictly correct[1], the collision term becomes:

$$
\left(\frac{\delta\vec{\mathbf{Y}}}{\delta t}\right)_c = -(1+Y^2)\left(\nu + \frac{1}{n}\left(\frac{\delta n}{\delta t}\right)_c\right)\vec{\mathbf{Y}} \tag{11}
$$

1 0-d example

In the absence of any spatial dependence, the equation takes on a much simpler form:

$$
\begin{aligned}
\frac{\partial Y}{\partial t} &- \quad vEX(2G(X)+1)\frac{\partial Y}{\partial u} = \\
& \quad vE\frac{Y}{X}(1+Y^2)\left(X^2\frac{\partial G(X)}{\partial u} + \frac{\partial H(X)}{\partial u}\right) \\
&+ \quad \frac{Y}{X}(1+Y^2)vE\left(\frac{X^2G(X)}{u} + (X^2G(X)+H(X))\frac{1}{n}\frac{\partial n}{\partial u}\right) \\
&- \quad Y(1+Y^2)\left(\nu + \frac{1}{n}\left(\frac{\delta n}{\delta t}\right)_c\right)
\end{aligned} \tag{12}
$$

which is to be solved, along with the usual first equation:

$$
\frac{\partial \eta}{\partial t} - \frac{\partial}{\partial u}(vEX\eta) = \left(\frac{\delta\eta}{\delta t}\right)_c \tag{13}
$$

where $\eta = vn$.

Alternately, Equation 12 can be written as:

$$
\begin{aligned}
\frac{\partial Y}{\partial t} &- \quad vEXK(X)\frac{\partial Y}{\partial u} = \\
& \quad \frac{Y}{X}(1+Y^2)vE\left(\frac{X^2G(X)}{u} + (F(X)-X^2)\frac{1}{n}\frac{\partial n}{\partial u}\right) \\
&- \quad Y(1+Y^2)\left(\nu + \frac{1}{n}\left(\frac{\delta n}{\delta t}\right)_c\right)
\end{aligned} \tag{14}
$$

where:

$$
K(X) = (2G(X)+1) + XG'(X) + \frac{1}{X}H'(X) \tag{15}
$$

If we define:

$$
F(X) = \frac{X}{\gamma} \tag{16}
$$

then:

$$K(X) = \frac{1}{X}(X^2G(X)+H(X))'+1 = \frac{1}{X}(F(X)-X^2)'+1 = \frac{1}{X}F'(X)-1 \quad (17)$$

The limits are:

$$\lim_{X\to 0} K(X) = \frac{3}{5} - \frac{48}{175}X^2 \qquad \lim_{X\to 1} K(X) = (1-X). \quad (18)$$

It helps to know that:

$$F(X) = \frac{X}{\gamma} = \frac{1}{3} + \frac{4}{5}X^2 - \frac{12}{175}X^4 \quad (19)$$

at small X.

1.1 Townsend discharge

An inhomogeneous 0-d problem with exponential spatial growth of fundamental quantities (n, Γ) can be treated with the unbounded anisotropy ("Y") formulation. The equations are:

$$\frac{\partial\eta}{\partial t} - \frac{\partial}{\partial u}(vEX\eta) = -\alpha vX\eta + \left(\frac{\delta\eta}{\delta t}\right)_c \quad (20)$$

where α is the Townsend coefficient, as defined in [1]. A similar term must be added to the anisotropic equation, derived from those spatial derivative terms which involve derivatives of fundamental quantities:

$$\begin{aligned}\frac{\partial Y}{\partial t} &- vEXK(X)\frac{\partial Y}{\partial u} = -\alpha v(1+Y^2)(F(X)-X^2)\frac{3}{2}\\ &+ (1+Y^2)^{\frac{3}{2}}vE\left(X^2\frac{G}{u} + (F(X)-X^2)\frac{1}{n}\frac{\partial n}{\partial u}\right)\\ &- Y(1+Y^2)\left(\nu + \frac{1}{n}\left(\frac{\delta n}{\delta t}\right)_c\right) \quad (21)\end{aligned}$$

1.2 Pulsed Townsend discharge

Another inhomogeneous 0-d problem assumes exponential temporal growth of fundamental quantities (n, Γ) and is treated with the unbounded anisotropy ("Y") formulation as follows:

$$\frac{\partial\eta}{\partial t} - \frac{\partial}{\partial u}(vEX\eta) = -\beta\eta + \left(\frac{\delta\eta}{\delta t}\right)_c \quad (22)$$

where β is the exponential growth rate, as determined by the net ionization rate. The anisotropic equation requires no such term:

$$\frac{\partial Y}{\partial t} - vEXK(X)\frac{\partial Y}{\partial u} =$$

$$
\begin{aligned}
& (1+Y^2)^{\frac{3}{2}} vE \left(X^2 \frac{G}{u} + (F(X) - X^2) \frac{1}{n} \frac{\partial n}{\partial u} \right) \\
- \quad & Y(1+Y^2) \left(\nu + \frac{1}{n} \left(\frac{\delta n}{\delta t} \right)_c \right)
\end{aligned} \tag{23}
$$

References

[1] E. A. Richley. Elliptic representation of the boltzmann equation with validity for all degrees of anisotropy. *Physical Review E*, 59(4):4533–4541, April 1999.

[2] E. A. Richley. Analysis of the low-pressure low-current dc positive column in neon. *Physical Review E*, 66(2), August 2002. Art. No. 026402.