Some thoughts on implementing Gauss' Law

Edward A. Richley

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It is desirable to create a convective flow representation of electromagnetic fields. This allows well-established time-dependent reactive flow models to accommodate the effects of space charge. This concept builds on a simpler case which I published in Phys Rev E v66 026402.

Beginning with Ampere's Law:

$$\frac{\partial \vec{\mathbf{E}}}{\partial t} = -\frac{1}{\epsilon_0}\vec{\mathbf{J}} + \frac{1}{\epsilon_0}\nabla \times \vec{\mathbf{H}} \tag{1}$$

Multiply Gauss' Law by a pseudo-velocity, $\vec{\mathbf{V}}_E$:

$$\vec{\mathbf{V}}_E \left(\nabla \cdot \vec{\mathbf{E}} = \frac{1}{\epsilon_0}\rho \right) \tag{2}$$

Add these two equations to get:

$$\frac{\partial \vec{\mathbf{E}}}{\partial t} + \vec{\mathbf{V}}_E \nabla \cdot \vec{\mathbf{E}} = \vec{\mathbf{V}}_E \left(\frac{1}{\epsilon_0}\rho \right) - \frac{1}{\epsilon_0}\vec{\mathbf{J}} + \frac{1}{\epsilon_0}\nabla \times \vec{\mathbf{H}} \tag{3}$$

Likewise, take Faraday's Law:

$$\frac{\partial \vec{\mathbf{H}}}{\partial t} = -\frac{1}{\mu}\nabla \times \vec{\mathbf{E}} \tag{4}$$

Multiply Gauss' Law for magnetism by a factor (magnetic pseudo-velocity) $\vec{\mathbf{V}}_H$:

$$\vec{\mathbf{V}}_H \left(\nabla \cdot \vec{\mathbf{H}} = 0 \right) \tag{5}$$

Add them to get:

$$\frac{\partial \vec{\mathbf{H}}}{\partial t} + \vec{\mathbf{V}}_H \nabla \cdot \vec{\mathbf{H}} = -\frac{1}{\mu}\nabla \times \vec{\mathbf{E}} \tag{6}$$

where $\vec{\mathbf{V}}_E = \vec{\mathbf{V}}_E(\vec{\mathbf{r}}, t)$ and $\vec{\mathbf{V}}_H = \vec{\mathbf{V}}_H(\vec{\mathbf{r}}, t)$ are chosen for stability and accuracy, without sacrificing time step size.

All the while, continuity is also being solved by the other transport equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\mathbf{J}} = 0 \tag{7}$$

The result is that "infinite" speed of propagation implied by Gauss' law is eliminated. Instead, characteristics at speeds of $|\vec{\mathbf{V}}_E|$ and $|\vec{\mathbf{V}}_H|$ are created. If both $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ are needed in the simulation, then there is an additional characteristic at $c = 1/\sqrt{\mu\epsilon_0}$. In many cases, it should be possible to re-define μ to make c much smaller, so as to not introduce a restrictive time-step limit. That is, $dt = dx/c$ can be made just a bit smaller than the fastest physcial process that must be followed.

The usual approximations for magneto quasi-statics will lead to non-physical results with this representation. In particular, if $\vec{\mathbf{H}}$ is ignored, or if a scalar electric potential is assumed, a proper steady-state value for $\vec{\mathbf{J}}$ can not be obtained in general. Some geometries (axially uniform cylindrical symmetry, for example) may allow for the elimination of one component of $\nabla \times \vec{\mathbf{H}}$ in Equation 3, which greatly simplifies those cases. However, in general, one might be able to re-define c (as described above) in order to accomodate the general case without introducing excessive computational burdens.

Stability of these equations is easily examined. For example, taking the divergence of Equation 3 gives:

$$\frac{\partial \mathcal{D}}{\partial t} + \nabla \cdot \vec{\mathbf{V}}_E \left(\mathcal{D} - \frac{\rho}{\epsilon_0} \right) + \vec{\mathbf{V}}_E \cdot (\nabla(\mathcal{D} - \frac{\rho}{\epsilon_0}) - \nabla \cdot \vec{\mathbf{J}} = 0 \tag{8}$$

where $\mathcal{D} = \nabla \cdot \vec{\mathbf{E}}$.

Linearization of this, with $\rho \to \rho + \tilde{\rho} e^{\alpha t + j\vec{\beta}\cdot\vec{\mathbf{r}}}$, etc. gives:

$$(\alpha + \nabla \cdot \vec{\mathbf{V}}_E + j\vec{\beta} \cdot \vec{\mathbf{V}}_E)(\frac{1}{\epsilon_o}\tilde{\rho} + \tilde{\mathcal{D}}) = 0 \tag{9}$$

The second term is just the linearization of Gauss' law, and would be present without the time-dependent modifications. The first term represents the growth factor introduced by the convective form of the equations. Any deviations from zero of the second term, due to round-off or truncation errors, are subject to growth at the rate described by the first term. In order to suppress the growth of these errors, it is necessary for α to have a negative real component when the first term is set equal to zero. The stability condition is thus:

$$\nabla \cdot \vec{\mathbf{V}}_E > 0 \tag{10}$$

This condition will ensure that numerical errors will not grow as a result of inclusion of the convective terms. By starting with initial conditions which satisfy Gauss' law, and by proper choice of a stable $\vec{\mathbf{V}}_E$, errors in the balance prescribed by Gauss' law will not be allowed to grow. A similar condition exists for the magnetic pseudo-velocity, $\vec{\mathbf{V}}_H$.